D. Bump's notes on the Poisson Summation Formula

[These are taken from p.1-2 of Bump's notes on the Riemann zeta function.]

Let f be an L^1 function on $\mathbb{R}$. Then the *Fourier transform* of f is defined by

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(y)\, e^{2\pi i x y}\, dy.$$

Proposition 1. *If* $f(x) = e^{-\pi x^2}$ *then* $f = \hat{f}$.

PROOF. Completing the square,

$$\hat{f}(x) = e^{-\pi y^2} \int_{-\infty}^{\infty} e^{-\pi(x+iy)^2}\, dy.$$

It is easy to justify moving the line of integration by Cauchy's theorem, and

$$\int_{-\infty}^{\infty} e^{-\pi(y+ix)^2}\, dx = \int_{-\infty}^{\infty} e^{-\pi y^2}\, dy.$$

This integral equals one, since it's square equals

$$\int_{\mathbb{R}^2} e^{-\pi(x^2+y^2)}\, dx\, dy,$$

which is easily evaluated by switching to polar coordinates. □

Proposition 2 (Poisson summation formula). *Suppose that* f *is a smooth function such that* $(1+x^2)^N f(x)$ *is bounded for all* N. *Then*

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n), \tag{1}$$

PROOF. We introduce the auxiliary function

$$F(x) = \sum_{-\infty}^{\infty} f(x+n).$$

This is absolutely and uniformly convergent, and is a smooth function with period 1. It therefore has a Fourier expansion:

$$F(x) = \sum_{-\infty}^{\infty} a_m\, e^{-2\pi i m x}.$$

$-\infty$

We evaluate the coefficients

$$a_m = \int_0^1 F(x)\, e^{2\pi i m x}\, dx = \int_0^1 \sum_{n=-\infty}^{\infty} f(x+n)\, e^{2\pi i m x}\, dx.$$

Since $e^{2\pi i m x} = e^{2\pi i m (x+n)}$, we may collapse the summation and the integration, and write

$$a_m = \int_{-\infty}^{\infty} f(x)\, e^{2\pi i m x}\, dx = \hat{f}(m).$$

Now

$$F(x) = \sum_{-\infty}^{\infty} \hat{f}(m)\, e^{2\pi i m x}.$$

Putting $x = 0$ we obtain (1). □

Although in the explanation above, the Poisson Summation Formula is presented as a straightforward result of Fourier analysis, it is possible to interpret it as a kind of simple "trace formula" on a torus. This is outlined in:

H.P. McKean, "Selberg's trace formula as applied to a compact Riemannian surface", *Communications in Pure and Applied Mathematics* **25** (1972) 225-246.

The PSF is seen in this context as relating the spectrum of the Laplacian on a torus to the lengths of its closed geodesics. This interpretation allows for generalisation, as the torus is a compact Riemann surface of genus 1. A general genus *n* compact Riemann surface also has a Laplacian spectrum and a set of shortest path lengths in each deformation class, and it turns out that these sets of values can be related according to an analogous formula, essentially the Selberg trace formula.

back to homepage
contact